Generalizations of Some Integral Numerical Calculus Methods

Dumitru EBÂNCĂ

Faculty of Mathematics and Computer Science,
Department of Computer Science,
University of Craiova, Romania
ebi@central.ucv.ro

Abstract. The article contains a class of formulae for approximation of integrals, formulae who are a generalization for some numerical integration formulae.
Math. Subjects Classification 2000: 68W25, 65-99
Keywords: integrals, methods for integral numerical calculus.

1 INTRODUCTION

In this article we introduce a method of determining generalised formulae of integral numerical calculus for integrals of the type presented below:

$$I = \int_a^b f(x)dx,$$

where $f : [a, b] \longrightarrow \mathbb{R}$ is a integrable function on the real interval $[a, b]$.

From these formulae we can deduce in particular classical methods or other new types of methods of integral numerical calculus.

2 METHOD PRESENTATION

Let $f : [a, b] \longrightarrow \mathbb{R}$ be an integrable function over the real interval $[a, b]$, $n \in \mathbb{N}^*$, $h = \dfrac{b-a}{n}$ and the equidistant points $x_k = a + kh$, $0 \le k \le n$.

Then we have:

$$\int_a^b f(x)dx = \sum_{k=0}^{n-1} \int_{x_k}^{x_{k+1}} f(x)dx \tag{1}$$

For the points:

$$x_k + i\tau,\ \tau = \frac{x_{k+1} - x_k}{m},\ 0 \le i \le s,$$

where m, $s \in \mathbb{N}^*$, $0 \leq k \leq n-1$, we take into consideration the Newton interpolation formulae:

$$f = p_{sk} + r_{sk}, \tag{2}$$

where ([1]):

$$p_{sk}(x) = f(x_k) + \sum_{i=1}^{s} \frac{\Delta_\tau^i f(x_k)}{i! \cdot \tau^i} \prod_{j=0}^{i-1} (x - x_k - j\tau) \tag{3}$$

In (3) by Δ_τ^i we denote the ascending differences of order i and step τ, finite differences defined by:

Definition 1.

$$\Delta_\tau f(x) = f(x+\tau) - f(x)$$
$$\Delta_\tau^i = \Delta_\tau \left(\Delta_\tau^{i-1}\right), \ i \geq 2$$

We assume that $f \in C^{(s+1)}([a,b])$. In this case, there is a point $c_k(x)$ placed between the points x, $x_k + i\tau$, $0 \leq i \leq s$, such that the remainder r_{sk} in the interpolation formulae (2) has the expresion ([1]):

$$r_{sk}(x) = \frac{1}{(s+1)!} f^{(s+1)}(c_k(x)) \prod_{j=0}^{s} (x - x_k - j\tau) \tag{4}$$

For $x = x_k + \tau t$ the formulae (3) and (4) become:

$$p_{sk}(x_k + \tau t) = f(x_k) + \sum_{i=1}^{s} \frac{\Delta_\tau^i f(x_k)}{i!} \prod_{j=0}^{i-1} (t - j) \tag{5}$$

$$r_{sk}(x_k + \tau t) = \frac{\tau^{s+1}}{(s+1)!} f^{(s+1)}(c_k(t)) \prod_{j=0}^{s} (t - j) \tag{6}$$

For $x = x_k + \tau t$ in the integration formulae from the right member of the formulae:

$$\int_{x_k}^{x_{k+1}} f(x)dx = \int_{x_k}^{x_{k+1}} p_{sk}(x)dx + \int_{x_k}^{x_{k+1}} r_{sk}(x)dx$$

we obtain:

$$\int_{x_k}^{x_{k+1}} f(x)dx = \tau \int_0^m p_{sk}(x_k + \tau t)dt + \tau \int_0^m r_{sk}(x_k + \tau t)dt \tag{7}$$

Using the formulae (5) we obtain:

$$\tau \int_0^m p_{sk}(x_k + \tau t)dt = \tau \left[m f(x_k) + \sum_{i=1}^{s} \frac{\Delta_\tau^i f(x_k)}{i!} \int_0^m \prod_{j=0}^{i-1} (t - j)\, dt \right] \tag{8}$$

Because $\tau = \frac{b-a}{mn}$, from (1), (7) and (8) we obtain:

$$\int_a^b f(x)dx = \frac{b-a}{mn}\sum_{k=0}^{n-1}\left[mf(x_k) + \sum_{i=1}^{s}\frac{\Delta_\tau^i f(x_k)}{i!}\int_0^m \prod_{j=0}^{i-1}(t-j)\,dt\right] + R(f) \quad (9)$$

where:

$$R(f) = \sum_{k=0}^{n-1}\tau\int_0^m r_{sk}(x_k+\tau t)dt. \quad (10)$$

From (6) and (10) we obtain for the remainder $R(f)$ the expresion:

$$R(f) = \frac{(b-a)^{s+2}}{m^{s+2}n^{s+2}(s+1)!}\sum_{k=0}^{n-1}\int_0^m f^{(s+1)}(c_k(t))\prod_{j=0}^{s}(t-j)\,dt \quad (11)$$

Numerical integration formulae (9), in which the remainder $R(f)$ has the expresion (11), represent a class of generalized methods of integral numerical calculus. For given values of the parameters s and m we obtain an unlimited number of methods for integrals approximation, with approximation errors of any order.

Remark 1. For $s = m = 2q+1$ the accuracy order of the formulae (9) is $2q+1$.

For $s = m+1 = 2q+1$ we have:

$$p_{s+1k}(x_k+\tau t) = p_{sk}(x_k+\tau t) + \frac{\Delta_\tau^{2q+1} f(x_k)}{(2q+1)!}\prod_{j=0}^{2q}(t-j) \quad (12)$$

Because:

$$\int_0^{2q}\prod_{j=0}^{2q}(t-j)dt = \int_{-q}^{q} t\prod_{j=1}^{q}(t^2-j^2)dt = 0$$

we have:

$$\int_0^{2q} p_{s+1k}(x_k+\tau t)dt = \int_0^{2q} p_{sk}(x_k+\tau t)dt$$

and so:

$$\int_{x_k}^{x_{k+1}} f(x)dx = \int_{x_k}^{x_{k+1}} p_{sk}(x)dx + \int_{x_k}^{x_{k+1}} r_{s+1k}(x)dx$$

So, in this case also the accuracy order is $2q+1$.

For these values $s = m+1 = 2q+1$ or $s = m = 2q+1$ we obtain Newton-Cotes formulae of closed types ([2],[3]).

Examples:

1. For $s = m = 1$ we obtain the trapeze formulae:

$$\int_a^b f(x)dx = \frac{b-a}{2n}\left[f(a) + f(b) + 2\sum_{k=1}^{n-1} f(x_k)\right] + R(f) \qquad (13)$$

where:

$$R(f) = -\frac{(b-a)^3}{12n^2} f^{(2)}(\xi),\ \xi \in [a,b]$$

2. For $s = m + 1 = 3$ we obtain the Simpson formulae:

$$\int_a^b f(x)dx = \frac{b-a}{6n}\left[f(a) + f(b) + 2\sum_{k=1}^{n-1} f(x_k) + 4\sum_{k=0}^{n-1} f\left(\frac{x_k + x_{k+1}}{2}\right)\right] + R(f) \qquad (14)$$

where:

$$R(f) = -\frac{(b-a)^5}{2880n^4} f^{(4)}(\xi),\ \xi \in [a,b]$$

3. For $s = m = 3$ we obtain the Newton formulae:

$$\int_a^b f(x)dx = \frac{b-a}{8n}\left[f(a) + f(b) + 2\sum_{k=1}^{n-1} f(x_k) + 3\sum_{k=0}^{n-1}\left(f\left(\frac{2x_k + x_{k+1}}{3}\right) + f\left(\frac{x_k + 2x_{k+1}}{3}\right)\right)\right] + R(f) \qquad (15)$$

where:

$$R(f) = -\frac{(b-a)^5}{6480n^4} f^{(4)}(\xi),\ \xi \in [a,b]$$

etc.

Remark 2. For values given to the parameters s and m, other than the ones presented in the previous remark, we obtain formulae of open type where a and/or b aren't between the points denoted by x_k.

Examples:

1. For $m = 4$ and $s = 2$ we obtain:

$$\int_a^b f(x)dx = \frac{b-a}{3n}\sum_{k=0}^{n-1}\left[2f(x_k) - 4f\left(\frac{3x_k + x_{k+1}}{4}\right) + 5f\left(\frac{x_k + x_{k+1}}{2}\right)\right] + R(f) \qquad (16)$$

where:

$$R(f) = \frac{(b-a)^4}{96n^3} f^{(3)}(\xi),\ \xi \in [a,b]$$

2. For $m = 4$ and $s = 3$ we obtain:

$$\int_a^b f(x)dx = \frac{b-a}{3n}\sum_{k=0}^{n-1}\left[2f\left(\frac{3x_k + x_{k+1}}{4}\right) - \right.$$
$$\left. - f\left(\frac{x_k + x_{k+1}}{2}\right) + 2f\left(\frac{x_k + 3x_{k+1}}{4}\right)\right] + R(f) \tag{17}$$

where:

$$R(f) = \frac{7(b-a)^5}{23040n^4} f^{(4)}(\xi),\ \xi \in [a, b]$$

References

[1] **Ebâncă D.** (1994) Methods of Numerical Computation. Ed. Sitech, Craiova (in Romanian).
[2] **Isaakson E., Keller H.B.**(1966) Analysis of Numerical Methods. John Wiley & Sons, New York.
[3] **Stoer J., Bulirsch R.** (1991) Introduction to Numerical Analysis. Second Edition. Springer-Verlag New York.